# Math 206A Lecture 14 Notes

Daniel Raban

October 29, 2018

## 1 $\mathcal{F}$-Vectors of Associahedra

### 1.1 $\mathcal{F}$-vectors of associahedra

Recall the GZK construction of the associahedron. Let $Q \subseteq R^2$ be a fixed convex $n$-gon, and let $\tau = \tau(Q)$ be the set of triangulations of $Q$. For every $T \in \tau$, let $f_T : V \to \mathbb{R}$ be

$$f_T(v) = \sum_{\substack{\triangle \subseteq T \\ v \in \triangle}} \operatorname{area}(\triangle)$$

Then $P_n = \operatorname{conv}(\{f_T : T \in \tau\})$.

**Theorem 1.1** (GZK). *$P_n$ has lattice $\alpha(P_n)$ isomorphic to the graph of all diagonal subdivisions of $Q$.*

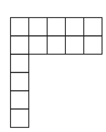
*Proof.* Here is a sketch. First, show $\dim(P_n) = n - 3$. Then show that $f_T$ are in convex positions. Finally show that $(f_T, f_{T'}) \in E(P_n) \iff T, T'$ differ by a flip. Then, use the Blind-Mani theorem. $\square$

We want to compute the $\mathcal{F}$-vector of $P_n$. Note that $f_k$ is the number of ways to place $n - 3 - k$ non-crossing diagonals in $Q$. This is sort of a generalization of the Catalan numbers because $f_0 = \binom{2n}{n}/(n+1)$. We can also see that $f_1 = (n-3)f_0/2$.

**Theorem 1.2.**

$$f_k = \frac{1}{n-k-2}\binom{n-3}{n-k-3}\binom{2n-k-4}{n-k-3}.$$

**Remark 1.1.** This is equal to the dimension of the representation of the symmetric group corresponding to the Young diagram



where the first 2 rows have $k$ boxes, and there are $2n$ rows.

Things are nicer with $g$-vectors, so let's work with those instead.

**Theorem 1.3.**

$$g_k = \frac{1}{n-2}\binom{n}{k}\binom{n}{k-1}.$$

*Proof.* Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a Morse function such that $\varphi(x_1, \ldots, x_n) = x_1 + \varepsilon x_2 + \varepsilon^2 x_3 + \cdots$. Look at the binary tree dual to $T$. Imagine entering from outside $Q$ and turning either left to right to travel along each edge of the binary tree. We can then denote the edges of the tree as left or right. Then flipping the edge corresponding to a left edge in the tree will increase the value of the Morse function $\varphi$ because it will change a diagonal connected to a larger labeled vertex to a smaller one. So $\mathrm{ind}_\varphi(T)$ is the number of left edges in the binary tree. So $g_n = h_k^\varphi$ is the number of binary trees with $n-2$ vertices.

Denote by $b(n,k)$ the number of binary trees on $n$ vertices with $k$ left edges. How can we count this? Start with a binary tree. There are $2n - (n-1) = n+1$ places to add an edge to increase the size of the tree. The number of left open places to put an edge is $(n-k)$, and the number of right open places is $(n+1)-(n-k)$. These relations give us a sort of Pascal's triangle for $b(n,k)$; we get $b(n+1,k) = kb(n,k-1) + (n-k)b(n,k)$. We can check the recurrence against the expression in the theorem. $\square$

**Remark 1.2.** In this specific example, the Dehn-Sommerville equations $g_k = g_{n-3-k}$ are just a consequence of the face that we can flip every edge of the triangulation to get another triangulation.

## 1.2 Narayana numbers

These numbers actually come up in a lot of places in combinatorics. They have a name.

**Definition 1.1.** The **Narayana numbers** are

$$N(n,k) := \frac{1}{n}\binom{n}{k}\binom{n}{k+1}.$$

**Proposition 1.1.**

$$\sum_{k=0}^{n-1} N(n,k) = \mathrm{Cat}(n) = \frac{1}{n+1}\binom{2n}{n}.$$

*Proof.* Each term in the sum on the left is the number of binary trees on $n$ vertices with $k$ left edges. The $n$-th Catalan number is the number of binary trees on $n$ vertices. $\square$